On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with the Dicyclic Group

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Abstract

A necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for the symmetry classes of tensors associated with the dicyclic group is given. In particular we apply these conditions to the generalized quaternion group, for which the dimensions of the symmetry classes of tensors are computed.

Keywords: Symmetry class of tensors, Orthogonal basis, Dicyclic group, 2-Adic valuation.

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1 Introduction

Let V be an m-unitary space. Let $\overset{n}{\otimes}V$ be the n-th tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in S_n there corresponds a unique linear operator $P(\sigma)$: $\overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let G be a subgroup of S_n and let $\mathrm{I}(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\Big\{T(G,\chi) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V \Big| \ T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)P(\sigma), \ \chi \in \mathrm{I}(G)\Big\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under the $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and it is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \ldots * v_n$ and it is called a *decomposable tensor*. It is well-known that

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)m^{c(\sigma)} \tag{1}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ (see [5]).

The inner product on V induces an inner product on $\overset{n}{\otimes} V$ whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \ldots * u_n \mid v_1 * \ldots * v_n \rangle = \frac{\chi(1)}{|G|} d_\chi^G(A)$$

where $A = [a_{ij}]_{n\times n} = [\langle u_i | v_j \rangle]_{n\times n}$ and $d_\chi^G(A) = \sum_{\sigma\in G} \chi(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$ is the *generalized matrix function.*

With respect to this inner product we have

$$\bigotimes^n V = \bigoplus_{\chi \in \mathrm{I}(G)} V_\chi^n(G)$$

which is an orthogonal direct sum.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \le \alpha_i \le m$, so α is a mapping from a set of n elements into a set of m elements. Then the group G acts on Γ_m^n by $\sigma.\alpha := (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$ where $\sigma \in G$ is a permutation on n letters and $\alpha \in \Gamma_m^n$ is a mapping from a set of n elements into a set of m elements. Therefore the action may be written as $\sigma.\alpha = \alpha\sigma^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{\sigma.\alpha |\ \sigma \in G\}$ be the *orbit* with representative α, also let G_α be the *stabilizer* of α, i.e., $G_\alpha = \{\sigma \in G |\ \sigma.\alpha = \alpha\}$. In this setting if $\alpha \in \Gamma_m^n$ and $\sigma \in G$, then we have $G_{\sigma.\alpha} = \sigma G_\alpha \sigma^{-1}$.

Let Δ be a system of distinct representatives of the orbits of G acting on Γ_m^n and define

$$\overline{\Delta} = \Big\{ \alpha \in \Delta |\ \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \Big\},$$

and let Ω be the union of those equivalence classes represented by elements of $\overline{\Delta}$.

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of V. Denote by e_α^* the tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$. We have

$$\langle e_\alpha^* \mid e_\beta^* \rangle = \begin{cases} \frac{\chi(1)}{|G|} \sum\limits_{\sigma \in G_\beta} \chi(\sigma\tau^{-1}) & \text{if} \quad \alpha = \tau.\beta \text{ for some } \tau \in G, \\ 0 & \text{if} \quad O(\alpha) \neq O(\beta), \end{cases}$$

in particular, taking the norm of e_α^*, with respect to the induced inner product, one easily obtains the condition $e_\alpha^* \neq 0$ if and only if $\alpha \in \Omega$.

For $\gamma \in \overline{\Delta}$, $V_\gamma^* = \langle e_{\sigma.\gamma}^* |\ \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$. It follows that

$$V_\chi^n(G) = \bigoplus_{\gamma \in \overline{\Delta}} V_\gamma^* \tag{2}$$

is an orthogonal direct sum. In [2] Freese proved that

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \chi(\sigma) \tag{3}$$

in particular, if χ is of degree one, then $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$.

If $\alpha = g.\gamma$ and $\beta = g'.\gamma$, then $gg'^{-1}.\beta = \alpha$, so if we set $\tau = gg'^{-1}$ and use the above formula for $\langle e_\alpha^* \mid e_\beta^* \rangle$, then we obtain

$$\langle e_{g.\gamma}^* \mid e_{g'.\gamma}^* \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in g' G_\gamma g^{-1}} \chi(\sigma). \tag{4}$$

An orthogonal basis of the form $\{e_\alpha^* | \ \alpha \in S\}$, where S is a subset of Γ_m^n, is called an *orthogonal basis of decomposable symmetrized tensors* for $V_\chi^n(G)$. By (2) $V_\chi^n(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if for all $\gamma \in \overline{\Delta}$, the orbital subspace V_γ^* has an orthogonal basis of decomposable symmetrized tensors. In particular, if χ is of degree one, since $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$, then V_γ^* has an orthogonal basis of decomposable symmetrized tensors for all $\gamma \in \overline{\Delta}$ which implies that $V_\chi^n(G)$ has such a basis.

Several papers are devoted in investigation of the existence of an orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$, see for example [10]. In [3] a necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ is given, where G is a cyclic or a dihedral group. In [6] it was claimed that if $\{e_1, \ldots, e_m\}$ is an orthogonal basis of V, then there exists a subset S of Γ_m^n such that the set $\{e_\alpha^* | \ \alpha \in S\}$ forms an orthogonal basis of $V_\chi^n(G)$ if and only if $\chi(1) = 1$. But later in [9] a counter-example to this claim was presented.

Now it is natural to consider a group G and an irreducible character $\chi \in \mathrm{I}(G)$ and obtain necessary and sufficient conditions for the existence of an orthogonal basis for $V_\chi^n(G)$. With respect to this we consider the dicyclic group which will be explained below.

In this paper we find a necessary and sufficient condition for the existence of an orthogonal basis of decomposable symmetrized tensors of symmetry classes of tensors associated with the dicyclic group. Throughout this paper all characters are considered over the complex field $\mathbb{C}$. We adopt notations from [7] in this paper.

2 Dicyclic Group

The group $T_{4n},\ n \geq 1$, generated by the elements $r,\ s$ such that $r^{2n} = 1,\ r^n = s^2$, $s^{-1}rs = r^{-1}$ is called the *dicyclic group* of degree n, i.e., $T_{4n} = \langle r,\ s|\ r^{2n} = 1,\ r^n = s^2, s^{-1}rs = r^{-1}\rangle$ (see [4]). This group is of order $4n$ and in [1] page 7 it is denoted by $\langle 2,\ 2,\ n\rangle$ and it is proved that $T_{4n} = \langle r,\ s|\ r^n = s^2 = (rs)^2\rangle$. In any case we have $T_{4n} = \{r^l,\ r^ls|\ 0 \leq l < 2n\}$.

By [4] T_{4n} has $n+3$ conjugacy classes which are

$$\begin{aligned}&\{1\},\ \{r^n\},\ \{r^k, r^{2n-k}\},\ 1 \leq k \leq n-1,\ \{r^{2k}s|\ 0 \leq k \leq n-1\},\\ &\{r^{2k+1}s|\ 0 \leq k \leq n-1\}\end{aligned}$$

and the character table of T_{4n} is indicated in Tables I and II.

Table I: The character table of T_{4n} where n is odd

$\lvert C_{T_{4n}}(g)\rvert$	$4n$	$4n$	$2n$	4	4
g	1	r^n	$r^k(1 \leq k \leq n-1)$	s	rs
ψ_1	1	1	1	1	1
ψ_2	1	-1	$(-1)^k$	i	$-i$
ψ_3	1	1	1	-1	-1
ψ_4	1	-1	$(-1)^k$	$-i$	i
χ_h $(1 \leq h \leq n-1)$	2	$2(-1)^h$	$2\cos(kh\pi/n)$	0	0

Table II: The character table of T_{4n} where n is even

$\lvert C_{T_{4n}}(g)\rvert$	$4n$	$4n$	$2n$	4	4
g	1	r^n	$r^k(1 \leq k \leq n-1)$	s	rs
ϕ_1	1	1	1	1	1
ϕ_2	1	1	$(-1)^k$	1	-1
ϕ_3	1	1	1	-1	-1
ϕ_4	1	1	$(-1)^k$	-1	1
χ_h $(1 \leq h \leq n-1)$	2	$2(-1)^h$	$2\cos(kh\pi/n)$	0	0

From the above tables we see that T_{4n} has four irreducible characters of degree 1 namely $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4$ if n is odd and $\phi_1,\ \phi_2,\ \phi_3,\ \phi_4$ if n is even and $n-1$ irreducible characters of degree 2 which are denoted by $\chi_h,\ 1 \leq h \leq n-1$.

By classical Cayley Theorem T_{4n} can be embed in S_{4n} and so we assume that T_{4n} is a subgroup of S_{4n}. In this case generators $r,\ s$ of T_{4n} as permutations on $4n$ letters are given by

$$\begin{aligned} r = &\ (1\ \ 2\ \ 3\ \ \ldots\ \ 2n)(2n+1\ \ 2n+2\ \ 2n+3\ \ \ldots\ \ 4n), \\ s = &\ (1\ \ 2n+1\ \ n+1\ \ 3n+1)(2\ \ 4n\ \ n+2\ \ 3n)(3\ \ 4n-1\ \ n+3\ \ 3n-1)\ldots \\ &\ (n-1\ \ 3n+3\ \ 2n-1\ \ 2n+3)(n\ \ 3n+2\ \ 2n\ \ 2n+2). \end{aligned}$$

In particular, the dicyclic group of degree 2^{n-1} is called the *generalized quaternion* group and denoted by $Q_{2^{n+1}}$, i.e., $Q_{2^{n+1}} = T_{4(2^{n-1})} = T_{2^{n+1}}$, and $\chi = \chi_h$, $1 \leq h \leq 2^{n-1}-1$, are characters of degree 2 for $Q_{2^{n+1}}$. In the following theorems we find the dimensions of the symmetry classes of tensors associated with the dicyclic group T_{4n}.

Theorem 1 *Let $G = T_{4n}$, n odd, and assume that V is an m-unitary space. Then considering G as a subgroup of the symmetric group on $4n$ letters we have the following, where $(2n,k)$ denotes the greatest common divisor of $2n$ and k.*

$$\dim V_{\psi_1}^{4n}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} m^{2(2n,k)} + 2nm^n\right],$$

$$\dim V_{\psi_2}^{4n}(G) = \frac{1}{4n}\left[m^{4n} - m^{2n} + 2\sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)}\right],$$

$$\dim V_{\psi_3}^{4n}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} m^{2(2n,k)} - 2nm^n\right],$$

$$\dim V_{\psi_4}^{4n}(G) = \frac{1}{4n}\left[m^{4n} - m^{2n} + 2\sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)}\right],$$

$$\dim V_{\chi_h}^{4n}(G) = \frac{1}{2n}\left[2m^{4n} + 2(-1)^h m^{2n} + 4\sum_{k=1}^{n-1} \cos(kh\pi/n) m^{2(2n,k)}\right],\ 1 \leq h \leq n-1.$$

Proof. Note that if π is a cycle of length a and $(k,a) = d$, then π^k has d cycles of length a/d and therefore $c(\pi^k) = d = (k,a)$. So $c(1) = 4n$, $c(r^n) = 2n$, $c(r^k) = 2(2n,k)$ and $c(s) = n$, where $c(\pi)$ denotes the number of cycles in the cycle structure of π including cycles of length one. Considering the cycle structures of r and s given above we obtain:

$$\begin{aligned} rs = \ & (1\ \ 2n+2\ \ n+1\ \ 3n+2)(2\ \ 2n+1\ \ n+2\ \ 3n+1)(3\ \ 4n\ \ n+3\ \ 3n)\ldots \\ & (n\ \ 3n+3\ \ 2n\ \ 2n+3), \end{aligned}$$

therefore $c(rs) = n$. Now using the character table of T_{4n} given in Table I the theorem holds by (1). □

Theorem 2 *Let $G = T_{4n}$, n even, and assume that V is an m-unitary space. Then considering G as a subgroup of the symmetric group on $4n$ letters we have the following, where $(2n,k)$ denotes the greatest common divisor of $2n$ and k.*

$$\dim V^{4n}_{\phi_1}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} m^{2(2n,k)} + 2nm^n\right],$$

$$\dim V^{4n}_{\phi_2}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)}\right],$$

$$\dim V^{4n}_{\phi_3}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} m^{2(2n,k)} - 2nm^n\right],$$

$$\dim V^{4n}_{\phi_4}(G) = \frac{1}{4n}\left[m^{4n} + m^{2n} + 2\sum_{k=1}^{n-1} (-1)^k m^{2(2n,k)}\right],$$

$$\dim V^{4n}_{\chi_h}(G) = \frac{1}{2n}\left[2m^{4n} + 2(-1)^h m^{2n} + 4\sum_{k=1}^{n-1} \cos(kh\pi/n) m^{2(2n,k)}\right], \quad 1 \leq h \leq n-1.$$

Proof. Similar to the proof of Theorem 1. □

The following lemma is useful in later considerations.

Lemma 1 *Let H be a subgroup of T_{4n}. Then there is a natural number k, $0 \leq k < 2n$, such that $H = \langle r^k \rangle$ or $\langle r^k \rangle \subsetneqq H$ and $H \cap \langle r \rangle = \langle r^k \rangle$. In the second case we have $|H| \geq 2|\langle r^k \rangle|$.*

Proof. By definition of T_{4n} we see that elements of T_{4n} are of the forms r^l or $r^l s$ where $0 \leq l < 2n$. If H is an arbitrary subgroup of T_{4n}, then $H \cap \langle r \rangle$ is a cyclic subgroup of $\langle r \rangle$ and therefore there is a natural number k, $0 \leq k < 2n$, such that $H \cap \langle r \rangle = \langle r^k \rangle$. The rest by our lemma follows immediately. □

3 On the Existence of Orthogonal Basis for the Symmetry Classes of Tensors Associated with the Dicyclic Group

Let $G = T_{4n},\ n \geq 1$, and V be an m-unitary space, with orthonormal basis $\{e_1, \ldots, e_m\}$. For $n = 1$, the dicyclic group T_4 is cyclic, $T_4 \cong \mathbb{Z}_4$, therefore all irreducible characters are of degree 1 and so $V_\chi^4(T_4)$ has an orthogonal basis of decomposable symmetrized tensors for all $\chi \in \mathrm{I}(T_4)$. Therefore we assume that $n \geq 2$. If $m = 1$, then $\dim \overset{4n}{\otimes} V = 1$, so $\dim V_\chi^{4n}(G) = 0$ or 1, therefore we don't have any problem about the existence of orthogonal basis of decomposable symmetrized tensors for $V_\chi^{4n}(G)$ for all $\chi \in \mathrm{I}(G)$, therefore we assume that $m \geq 2$.

For irreducible characters of T_{4n} of degree 1, $\psi_i,\ \phi_i,\ 1 \leq i \leq 4,\ V_{\psi_i}^{4n}(T_{4n})$ and $V_{\phi_i}^{4n}(T_{4n})$ have an orthogonal basis of decomposable symmetrized tensors and so we don't deal with the ψ_i's and ϕ_i's.

Therefore we investigate the problem for irreducible characters of degree 2 of T_{4n}, i.e., $\chi_h, 1 \leq h \leq n-1$, which are given by

$$\chi_h(r^k) = 2\cos\frac{kh\pi}{n}\ , \quad \chi_h(r^k s) = 0, \qquad 0 \leq k < 2n.$$

Lemma 2 *Suppose $n \geq 2,\ 1 \leq h \leq n-1,\ 0 \leq k < 2n$. Let $l = \big(2n/(2n,k)\big)$, where $(2n,k)$ denotes the greatest common divisor of $2n$ and k. Then we have*

$$\sum_{t=1}^{l} \cos\frac{tkh\pi}{n} = \begin{cases} l & \text{if} \quad kh \overset{2n}{\equiv} 0, \\ 0 & \text{if} \quad kh \overset{2n}{\not\equiv} 0. \end{cases}$$

Proof. It is straightforward. □

Lemma 3 *Suppose $G = T_{4n},\ n \geq 2$, and $\chi = \chi_h,\ 1 \leq h \leq n-1$. Let H be any subgroup of T_{4n}, i.e., $H = \langle r^k \rangle$ or $\langle r^k \rangle \lneqq H$ and $H \cap \langle r \rangle = \langle r^k \rangle$, for some $k,\ 0 \leq k < 2n$. If $l = \big(2n/(2n,k)\big)$, where $(2n,k)$ denotes the greatest common divisor of $2n$ and k, then we have*

$$\sum_{g \in H} \chi(g) = \begin{cases} 2l & \text{if} \quad kh \overset{2n}{\equiv} 0, \\ 0 & \text{if} \quad kh \overset{2n}{\not\equiv} 0. \end{cases}$$

Proof. We have $o(r^k) = \big(2n/(2n,k)\big) = l$, so $H = \{r^k, r^{2k}, \dots, r^{lk}\}$ or $\{r^k, r^{2k}, \dots, r^{lk}\} \lneqq H$ and $H \cap \langle r \rangle = \{r^k, r^{2k}, \dots, r^{lk}\}$. But χ vanishes outside $\langle r \rangle$, therefore by Lemma 2 we have

$$\sum_{g \in H} \chi(g) = \sum_{t=1}^{l} \chi(r^{tk}) = 2 \sum_{t=1}^{l} \cos \frac{tkh\pi}{n} = \begin{cases} 2l & \text{if} \quad kh \overset{2n}{\equiv} 0, \\ 0 & \text{if} \quad kh \overset{2n}{\not\equiv} 0. \end{cases}$$

□

Lemma 4 *Let* $G = T_{4n},\ n \geq 2$, *and* $\chi = \chi_h,\ 1 \leq h \leq n-1$. *Then for* $\gamma \in \overline{\Delta}$, *we have* $G_\gamma = \langle r^k \rangle$ *or* $\langle r^k \rangle \lneqq G_\gamma$ *and* $G_\gamma \cap \langle r \rangle = \langle r^k \rangle$, *for some* $k,\ 0 \leq k < 2n$, *where* $kh \overset{2n}{\equiv} 0$. *In particular, if* $\langle r^k \rangle \lneqq G_\gamma$, *then we have* $|G_\gamma| \geq 2|\langle r^k \rangle|$.

Proof. G_γ is a subgroup of G so by Lemma 1, $G_\gamma = \langle r^k \rangle$ or $\langle r^k \rangle \lneqq G_\gamma$ and $G_\gamma \cap \langle r \rangle = \langle r^k \rangle$, for some $k,\ 0 \leq k < 2n$. In particular if $\langle r^k \rangle \lneqq G_\gamma$, then $|G_\gamma| \geq 2|\langle r^k \rangle|$. But by Lemma 3 if $kh \overset{2n}{\not\equiv} 0$, then $\sum_{g \in G_\gamma} \chi(g) = 0$ and so $\gamma \notin \overline{\Delta}$. This contradiction show that $kh \overset{2n}{\equiv} 0$. □

Lemma 5 *Let* $n \geq 2$ *and* $1 \leq h \leq n-1$. *Then there exist* $t,\ t',\ 0 \leq t,\ t' < 2n$, *such that* $\cos\big((t-t')h\pi/n\big) = 0$ *if and only if* $\nu_2(h/n) < 0$, *where* ν_2 *is the 2-adic valuation.*

Proof. It is straightforward. For the definition of p-adic valuation we refer the reader to [8]. □

Lemma 6 *Suppose* $G = T_{4n},\ n \geq 2$, *and let* $\chi = \chi_h,\ 1 \leq h \leq n-1$. *Let* $\gamma \in \overline{\Delta}$ *and suppose that* G_γ *is of the form* $G_\gamma = \langle r^k \rangle$, *where* $0 \leq k < 2n,\ kh \overset{2n}{\equiv} 0$. *If* $\nu_2(h/n) < 0$, *where* ν_2 *is the 2-adic valuation, then the orbital subspace* V_γ^* *has an orthogonal basis of decomposable symmetrized tensors.*

Proof. We have $o(r^k) = \big(2n/(2n,k)\big) = l$, so $G_\gamma = \{r^k, r^{2k}, \dots, r^{lk}\}$ and therefore by (3) and Lemma 3

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) = \frac{2}{l}(2l) = 4.$$

Now for all $g,\ g' \in G$ we have

$$g' G_\gamma g^{-1} = \begin{cases} \{r^{k+b-a}, r^{2k+b-a}, \dots, r^{lk+b-a}\} & \text{if} \quad g = r^a,\ g' = r^b, \\ \{r^{k+n+a+b}s, r^{2k+n+a+b}s, \dots, r^{lk+n+a+b}s\} & \text{if} \quad g = r^a s,\ g' = r^b, \\ \{r^{-k+b-a}, r^{-2k+b-a}, \dots, r^{-lk+b-a}\} & \text{if} \quad g = r^a s,\ g' = r^b s. \end{cases}$$

For $g = r^a,\ g' = r^b$ by (4) we have

$$\begin{aligned}
\langle e^*_{g.\gamma} | e^*_{g'.\gamma}\rangle &= \tfrac{\chi(1)}{|G|} \sum_{\sigma \in g' G_\gamma g^{-1}} \chi(\sigma) \\
&= \tfrac{2}{4n} \sum_{t=1}^{l} \chi(r^{tk+b-a}) \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \tfrac{(tk+b-a)h\pi}{n} \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \left(\tfrac{tkh\pi}{n} + \tfrac{(b-a)h\pi}{n}\right) \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \tfrac{(b-a)h\pi}{n} \\
&= \tfrac{l}{n} \cos \tfrac{(b-a)h\pi}{n}.
\end{aligned}$$

If $g = r^a s$ and $g' = r^b$ or $g = r^a s$ and $g' = r^b s$ then by the same computation we obtain $\langle e^*_{g.\gamma} | e^*_{g'.\gamma}\rangle = 0$ or $\langle e^*_{g.\gamma} | e^*_{g'.\gamma}\rangle = (l/n)\cos\big((b-a)h\pi/n\big)$, respectively. Therefore

$$\langle e^*_{g.\gamma} | e^*_{g'.\gamma}\rangle = \begin{cases} \tfrac{l}{n} \cos \tfrac{(b-a)h\pi}{n} & \text{if} \quad g = r^a,\ g' = r^b, \\ 0 & \text{if} \quad g = r^a s,\ g' = r^b, \\ \tfrac{l}{n} \cos \tfrac{(b-a)h\pi}{n} & \text{if} \quad g = r^a s,\ g' = r^b s. \end{cases}$$

Since by the assumption $\nu_2(\frac{h}{n}) < 0$, hence by Lemma 5 there exist $t,\ t',\ 0 \le t,\ t' < 2n$, such that $\cos\big((t-t')h\pi/n\big) = 0$. Let $S = \{r^t.\gamma, r^{t'}.\gamma, r^t s.\gamma, r^{t'} s.\gamma\} \subset \Gamma^{4n}_m$, then by the above computation, we have $\langle e^*_\alpha | e^*_\beta\rangle = 0$ for all $\alpha,\ \beta \in S,\ \alpha \ne \beta$. But $\dim V^*_\gamma = 4$, so $\{e^*_\alpha |\ \alpha \in S\}$ is an orthogonal basis of decomposable symmetrized tensors for V^*_γ. □

Lemma 7 *Suppose $G = T_{4n},\ n \ge 2$, and $\chi = \chi_h,\ 1 \le h \le n-1$. Let $\gamma \in \overline{\Delta}$ be such that $\langle r^k\rangle \subsetneqq G_\gamma$ and $G_\gamma \cap \langle r\rangle = \langle r^k\rangle$, where $0 \le k < 2n$, $kh \overset{2n}{\equiv} 0$. If $\nu_2(\frac{h}{n}) < 0$, where ν_2 is 2-adic valuation, then the orbital subspace V^*_γ has an orthogonal basis of decomposable symmetrized tensors.*

Proof. We have $o(r^k) = \big(2n/(2n,k)\big) = l$, so $\{r^k, r^{2k}, \ldots, r^{lk}\} \subsetneqq G_\gamma$ and $G_\gamma \cap \langle r\rangle = \{r^k, r^{2k}, \ldots, r^{lk}\}$. Note that, by Lemma 4, in this case we have $|G_\gamma| \ge 2l$, and therefore by (3) and Lemma 3

$$\dim V^*_\gamma = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) \le \frac{2}{2l}(2l) = 2$$

so $\dim V_\gamma^* = 1$ or $\dim V_\gamma^* = 2$. If $\dim V_\gamma^* = 1$, then we don't have any problem about the existence of orthogonal basis of decomposable symmetrized tensors for V_γ^*, therefore we assume that $\dim V_\gamma^* = 2$. For $g = r^a$, $g' = r^b$ we have $\{r^{k+b-a}, r^{2k+b-a}, \ldots, r^{lk+b-a}\} \lneqq g'G_\gamma g^{-1}$ and $g'G_\gamma g^{-1} \cap \langle r \rangle = \{r^{k+b-a}, r^{2k+b-a}, \ldots, r^{lk+b-a}\}$ therefore by (4) we have

$$\begin{aligned}
\langle e^*_{g.\gamma} | e^*_{g'.\gamma} \rangle &= \tfrac{\chi(1)}{|G|} \sum_{\sigma \in g'G_\gamma g^{-1}} \chi(\sigma) \\
&= \tfrac{2}{4n} \sum_{t=1}^{l} \chi(r^{tk+b-a}) \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \tfrac{(tk+b-a)h\pi}{n} \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \left(\tfrac{tkh\pi}{n} + \tfrac{(b-a)h\pi}{n} \right) \\
&= \tfrac{1}{n} \sum_{t=1}^{l} \cos \tfrac{(b-a)h\pi}{n} \\
&= \tfrac{l}{n} \cos \tfrac{(b-a)h\pi}{n}.
\end{aligned}$$

Since by assumption $\nu_2(\frac{h}{n}) < 0$, hence by Lemma 5 there exist $t,\ t',\ 0 \le t,\ t' < 2n$, such that $\cos\big((t-t')h\pi/n\big) = 0$, therefore by the above computation $\langle e^*_{r^t.\gamma} | e^*_{r^{t'}.\gamma} \rangle = 0$. For $S = \{r^t.\gamma, r^{t'}.\gamma\} \subset \Gamma_m^{4n}$, since $\dim V_\gamma^* = 2$, so $\{e^*_\alpha |\ \alpha \in S\}$ is an orthogonal basis of decomposable symmetrized tensors for V_γ^*. $\square$

Theorem 3 *Let $G = T_{4n}$, $n \ge 2$, and $\chi = \chi_h$, $1 \le h \le n-1$, $\dim V = m \ge 2$. Then $V_\chi^{4n}(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if $\nu_2(h/n) < 0$, where ν_2 is 2-adic valuation.*

Proof. Assume $V_\chi^{4n}(G)$ has an orthogonal basis of decomposable symmetrized tensors, therefore by (2) for all $\gamma \in \overline{\Delta}$, the orbital subspace V_γ^* has an orthogonal basis of decomposable symmetrized tensors, in particular for $\gamma = (1, 2, 2, \ldots, 2)$. Note that in this case $G_\gamma = \{1\}$, and $\sum_{g \in G_\gamma} \chi(g) = 2 \neq 0$, so $\gamma \in \overline{\Delta}$. For all $g,\ g' \in G$, we have

$$g'G_\gamma g^{-1} = \begin{cases} \{r^{b-a}\} & \text{if} \quad g = r^a,\ g' = r^b, \\ \{r^{n+a+b}s\} & \text{if} \quad g = r^a s,\ g' = r^b, \\ \{r^{b-a}\} & \text{if} \quad g = r^a s,\ g' = r^b s. \end{cases}$$

Therefore by (4) we have

$$\langle e^*_{g.\gamma} | e^*_{g'.\gamma} \rangle = \begin{cases} \frac{1}{n}\cos\frac{(b-a)h\pi}{n} & \text{if} \quad g = r^a, \ g' = r^b, \\ 0 & \text{if} \quad g = r^a s, \ g' = r^b, \\ \frac{1}{n}\cos\frac{(b-a)h\pi}{n} & \text{if} \quad g = r^a s, \ g' = r^b s. \end{cases}$$

But by (3)

$$\dim V^*_\gamma = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g) = \frac{2}{1}(2) = 4.$$

By the above computation if there are 4 decomposable symmetrized tensors for which any distinct pair are mutually orthogonal, then there should exist $t, \ t', \ 0 \leq t, \ t' < 2n$, such that $\cos\big((t-t')h\pi/n\big) = 0$. Therefore by Lemma 5 we obtain $\nu_2(h/n) < 0$.

Conversely assume $\nu_2(h/n) < 0$, then by Lemmas 4, 6 and 7, for all $\gamma \in \overline{\Delta}$, V^*_γ has the orthogonal basis of decomposable symmetrized tensors, and therefore by (2) so does $V^{4n}_\chi(G)$. □

Corollary 1 *Let* $G = T_{4n}, \ n \geq 2$ *is odd, and* $\chi = \chi_h, \ 1 \leq h \leq n-1, \ \dim V = m \geq 2$. *Then* $V^{4n}_\chi(G)$ *does not have an orthogonal basis of decomposable symmetrized tensors.*

Proof. Since n is odd, therefore $\nu_2(h/n) \geq 0$, and by Theorem 3 the corollary holds. □

Theorem 4 *Let* $G = Q_{2^{n+1}}, \ n \geq 2$, *the generalized quaternion group, and* $\chi = \chi_h, 1 \leq h \leq 2^{n-1} - 1, \ \dim V = m \geq 2$. *Then* $V^{2^{n+1}}_\chi(G)$ *has an orthogonal basis of decomposable symmetrized tensors.*

Proof. Note that $G = Q_{2^{n+1}} = T_{4(2^{n-1})}$, and since $1 \leq h \leq 2^{n-1} - 1$ therefore $\nu_2(h/2^{n-1}) < 0$ and by Theorem 3, this result follows. □

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